On reciprocal equivalence of Stäckel systems

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Abstract

In this paper we ivestigate Stäckel transforms between different classes of parameter-dependent Stäckel separable systems of the same dimension. We show that the set of all Stäckel systems of the same dimension splits to equivalence classes so that all members within the same class can be connected by a single Stäckel transform. We also give an explicit formula relating solutions of two Stäckel-related systems. These results show in particular that any two geodesic Stäckel systems are Stäckel equivalent in the sense that it is possible to transform one into another by a single Stäckel transform. We also simplify proofs of some known statements about multiparameter Stäckel transform.

Keywords and phrases: Hamiltonian systems, completely integrable systems, Stäckel systems, Hamilton-Jacobi theory, Stäckel transform

1 Introduction

Stäckel transform is a functional transform that transforms a given Liouville integrable system into a new integrable system on the same Poisson manifold. It was first described by J. Hietarinta et al in [1] (where it was called the coupling-constant metamorphosis) and developed in [2]. It has been applied in [3, 4, 5, 6] for the purpose of classification of superintegrable systems in conformally flat spaces. In [7, 8] the author described Stäckel transform as a canonical transformation on an extended phase space. Applied to a Stäckel separable system, this transformation yields a new Stäckel separable system, which explains its name.

Originally, only one coupling constant, entering linearly in one of the Hamiltonians of the system, was used. In paper [9] a multiparameter generalization of Stäckel transform has been introduced. This generalization allows for a nonlinear dependence of Hamiltonians of the system on several coupling parameters, thus much enlarging the class of admissible Stäckel transforms. Also, this generalized transform results in a class of reciprocal transformations that has been applied in [10] for analyzing weakly-nonlinear semi-Hamiltonian hydrodynamic-type systems. This indicates that Stäckel transform is a useful tool for studying various integrable systems. It can also be generalized for studying systems of ODE's of evolutionary type with integrals of motion, see [11].

In this paper we use the approach developed in [9] and further in [12] to show that all Stäckel systems of the same dimension n can be split into equivalence classes such that every two members of the same class are Stäckel equivalent in the sense that there always exists a single n-parameter Stäckel transform between arbitrary two such systems. In order to do this we consider Stäckel transforms inside given classes

of Stäckel systems, a problem not considered in previous papers. We also give an explicit, compact form of this transform, making the formulas more transparent then these in [9] and [12]. Also, we present a corresponding reciprocal transform between solutions of these Stäckel-related systems confined to proper submanifolds of the phase space. We also clarify and in some cases also repair a number of formulas and simplify proofs of a number of statements given in [9]. Two extensive examples are given at the end of the paper.

2 General Stäckel transform

In this section we present some facts about multiparameter Stäckel transform. Consider a manifold M equipped with a Poisson tensor Π. Denote the space of all smooth functions on M by $C^\infty(M)$. The mapping $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by $\{f,g\}_\Pi = (df, \Pi dg)$ (where $(\cdot,\cdot)$ is the dual map between cotangent and tangent spaces) is called Poisson bracket and it turns $C^\infty(M)$ into a Lie algebra. Suppose we have r functions (later: Hamiltonians) $h_i : M \to R$ on M, each depending on $k \leq r$ parameters $\alpha_1, \ldots, \alpha_k$ so that

$$h_i = h_i(x, \alpha_1, \ldots, \alpha_k), \quad i = 1, \ldots, r, \tag{1}$$

where $x \in M$. Let us now from r functions in (1) choose k functions h_{s_i}, $i = 1, \ldots, k$, where $\{s_1, \ldots, s_k\} \subset \{1, \ldots, r\}$. Assume also that

$$\det\left(\partial h_{s_i}/\partial \alpha_j\right) \neq 0. \tag{2}$$

The condition (2) means that the system of equations

$$h_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \quad i = 1, \ldots, k, \tag{3}$$

(where $\tilde{\alpha}_i$ is another set of k free parameters, or values of Hamiltonians h_{s_i}) involving the functions h_{s_i} can be locally solved for the parameters α_i yielding

$$\alpha_i = \widetilde{h}_{s_i}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \quad i = 1, \ldots, k, \tag{4}$$

where the right hand sides of these solutions define k new functions $\widetilde{h}_{s_i}$ on M, each depending on k parameters $\tilde{\alpha}_i$. Finally, let us define $r-k$ functions $\widetilde{h}_i$ with $i = 1, \ldots, r$ and such that $i \notin \{s_1, \ldots, s_k\}$ by - in accordance with (4) - substituting $\widetilde{h}_{s_i}$ instead of α_i in h_i for $i \notin \{s_1, \ldots, s_k\}$:

$$\widetilde{h}_i = h_i|_{\alpha_1 \to \widetilde{h}_{s_1}, \ldots, \alpha_k \to \widetilde{h}_{s_k}}, \quad i = 1, \ldots, r, \;\; i \notin \{s_1, \ldots, s_k\}. \tag{5}$$

Definition 1 *The functions $\widetilde{h}_i = \widetilde{h}_i(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$, $i = 1, \ldots, r$, defined through (4) and (5) are called the (generalized) Stäckel transform of the functions (1) with respect to the indices $\{s_1, \ldots, s_k\}$ (or with respect to the functions $h_{s_1}, \ldots h_{s_k}$).*

Note that unless we extend the manifold M this operation can in general not be obtained by any coordinate change of variables. It is also easy to see that if we perform again the Stäckel transform on the functions $\widetilde{h}_i$ with respect to $\widetilde{h}_{s_i}$ we will receive back the functions h_i in (1). In this sense the Stäckel transform is a reciprocal transform. Note also that neither k nor r are related to the dimension of the manifold M.

Example 2 *The simplest situation occurs when $k = r = 1$. Consider, after [1], the Fokas-Lagerström potential on the four-dimensional phase space M with coordinates (x, y, p_x, p_y) :*

$$h = \frac{1}{2}(p_x^2 + p_y^2) - \frac{2}{3}\alpha(xy)^{-2/3}$$

Solving the equation $h = \tilde{\alpha}$ with respect to the only parameter α (called in [1] a coupling constant) one obtains

$$\alpha = \frac{3}{4}(xy)^{2/3}(p_x^2 + p_y^2) - \frac{3}{2}(xy)^{2/3}\widetilde{\alpha} \equiv \widetilde{h}$$

which can be shown [1] to be equivalent to the axially symmetric potential ρ^4.

Stäckel transform has two important properties that make it well suited for study of integrable systems: as we will see in Theorem 3, it preserves functional independence and it also preserves involutivity with respect to the Poisson tensor Π. Moreover, as it will also be demonstrated in this paper, it maps a Stäckel separable system into a new Stäckel separable system which explains the name of this transformation.

In the special but nonetheless important for this paper case when functions (1) depend linearly on parameters α_i it is possible to write down the Stäckel transform explicitly. Suppose therefore for the moment that the functions in (1) have the form

$$h_i = H_i + \sum_{j=1}^{k} \alpha_j H_i^{(j)}, \quad i = 1, \ldots, r. \tag{6}$$

The equations (3) defining the first part of the Stäckel transform take then the form of a system of k linear equations in k unknowns $\alpha_1, \ldots, \alpha_k$

$$H_{s_i} + \sum_{j=1}^{k} \alpha_j H_{s_i}^{(j)} = \tilde{\alpha}_i, \quad i = 1, \ldots, k,$$

with the Cramer solution for $\alpha_i = \widetilde{h}_{s_i}$ of the form:

$$\widetilde{h}_{s_i} = \det W_i / \det W, \tag{7}$$

where

$$W = \begin{vmatrix} H_{s_1}^{(1)} & \cdots & H_{s_1}^{(k)} \\ \vdots & \ddots & \vdots \\ H_{s_k}^{(1)} & \cdots & H_{s_k}^{(k)} \end{vmatrix}$$

is the $k \times k$ matrix $\det\left(\partial h_{s_i}/\partial \alpha_j\right)$ given in (2) (so that $\det W \neq 0$) and where W_i are obtained from W by replacing $H_{s_j}^{(i)}$ in the i-th column by $\tilde{\alpha}_j - H_{s_j}$ for all $j = 1, \ldots, k$. The second part of the transformation, i.e. formulas (5), reads now

$$\widetilde{h}_i = H_i + \sum_{j=1}^{k} \widetilde{h}_{s_j} H_i^{(j)}, \quad i = 1, \ldots, r, \quad i \notin \{s_1, \ldots, s_k\}$$

where $\widetilde{h}_{s_i}$ are given by (7). For $k = 1$ the above transformation reproduces the original Stäckel transform presented in [1] and [2].

3 Stäckel transform for completely integrable systems

Let us now discuss the Stäckel transform and the corresponding reciprocal transform between two Liouville integrable systems. Suppose therefore that $\dim M = 2n$ and that we have exactly n (so that $r = n$ now) functionally independent functions (Hamiltonians)

$$h_i = h_i(x, \alpha_1, \ldots, \alpha_k), \quad i = 1, \ldots, n$$

that depend on $k \leq n$ parameters α_i and that are for all values of α_i in involution with respect to a nondegenerate Poisson bracket Π: $\{h_i, h_j\}_\Pi = 0$ for all i, j. These functions yield n commuting Hamiltonian systems on M:

$$\frac{dx}{dt_i} = \Pi dh_i \equiv X_i, \; i = 1, \ldots, n \tag{8}$$

so that X_i are n commuting Hamiltonian vector fields on M. Consider now a new set of n functions (Hamiltonians) $\widetilde{h}_i$ obtained from h_i by a Stäckel transform performed with respect to $h_{s_1}, \ldots, h_{s_k}$ (recall that $k \leq n$). These functions define a set of Hamiltonian flows on M, the vector fields of which are given by

$$\frac{dx}{d\widetilde{t}_i} = \Pi d\widetilde{h}_i \equiv \widetilde{X}_i, \; i = 1, \ldots, n. \tag{9}$$

Let us now consider the relation between the Hamiltonian systems (8) and (9). As it follows from (3) and (4) the following identity is valid on M and for all values of parameters $\widetilde{\alpha}_i$:

$$h_{s_i}(x, \widetilde{h}_{s_1}(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n), \ldots, \widetilde{h}_{s_k}(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)) \equiv \widetilde{\alpha}_i, \ \ i = 1, \ldots, k \tag{10}$$

Moreover, the second part of the transformation, i.e. formula (5) can be written as the following identity on M (valid again for all values of $\widetilde{\alpha}_i$):

$$\widetilde{h}_i(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n) \equiv h_i(x, \widetilde{h}_{s_1}(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n), \ldots, \widetilde{h}_{s_k}(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)), \quad i = 1, \ldots, n, \ \ i \notin \{s_1, \ldots, s_k\} \tag{11}$$

Differentiating (10) with respect to x_p we find that

$$dh_{s_i} = -\sum_{j=1}^{k} \frac{\partial h_{s_i}}{\partial \alpha_j} d\widetilde{h}_{s_j}, \quad i = 1, \ldots, k \tag{12}$$

while differentiation of (11) gives

$$dh_i = d\widetilde{h}_i - \sum_{j=1}^{k} \frac{\partial h_i}{\partial \alpha_j} d\widetilde{h}_{s_j}, \quad i = 1, \ldots, n, \quad i \notin \{s_1, \ldots, s_k\} \ . \tag{13}$$

The transformation (12)-(13) can be written in a matrix form as

$$dh = A d\widetilde{h} \tag{14}$$

where we denote $dh = (dh_1, \ldots, dh_n)^T$ and $d\widetilde{h} = (d\widetilde{h}_1, \ldots, d\widetilde{h}_n)^T$ and where the $n \times n$ matrix A is given by

$$\begin{aligned}
A_{ij} &= \delta_{ij} \text{ for } i \notin \{s_1, \ldots, s_k\}, \, j \notin \{s_1, \ldots, s_k\} \\
A_{is_j} &= -\frac{\partial h_i}{\partial \alpha_j}, \ \text{ for } i \notin \{s_1, \ldots, s_k\}, \, j = 1, \ldots, k \\
A_{s_i j} &= 0 \text{ for } i = 1, \ldots, k, \quad j \notin \{s_1, \ldots, s_k\} \\
A_{s_i s_j} &= -\frac{\partial h_{s_i}}{\partial \alpha_j} \text{ for } i, j = 1, \ldots, k.
\end{aligned}$$

¿From the structure of the matrix A it follows that

$$\det A = \pm \det \left(\frac{\partial h_{s_i}}{\partial \alpha_j} \right)$$

so that $\det A \neq 0$ due to the assumption (2). Thus, the relation (14) can be inverted yielding $d\widetilde{h} = A^{-1} dh$. This leads to an important theorem [9], mentioned in Section 2.

Theorem 3 *1. If the functions h_i are functionally independent then so are $\widetilde{h}_i$. 2. If the functions h_i are in involution with respect to the Poisson tensor Π (for all values of α_i), then the functions $\widetilde{h}_i$ are also in involution with respect to Π for all values of $\widetilde{\alpha}_i$.*

Proof. 1. Assume that h_i are functionally independent. It means that dh_i are linearly independent at every point of M. Then, by (14) and by the fact that $\det A \neq 0$ the differentials $d\widetilde{h}_i$ also are linearly independent and thus $\widetilde{h}_i$ are functionally independent on M. 2. Assume $\{h_i, h_j\}_\Pi = 0$ for all $i, j = 1, \ldots, n$. Then

$$\begin{aligned}
\left\{\widetilde{h}_i, \widetilde{h}_j\right\}_\Pi &= \left(d\widetilde{h}_i, \Pi d\widetilde{h}_j\right) = \left(\sum_{l_1=1}^{n} \left(A^{-1}\right)_{il_1} dh_{l_1}, \Pi \sum_{l_2=1}^{n} \left(A^{-1}\right)_{jl_2} dh_{l_2} \right) \\
&= \sum_{l_1, l_2=1}^{n} \left(A^{-1}\right)_{il_1} \left(A^{-1}\right)_{jl_2} (dh_{l_1}, \Pi dh_{l_2}) = \sum_{l_1, l_2=1}^{n} \left(A^{-1}\right)_{il_1} \left(A^{-1}\right)_{jl_2} \{h_{l_1}, h_{l_2}\}_\Pi = 0
\end{aligned}$$

■

Theorem 3 implies that the system (9) is again Liouville integrable so that Stäckel transform maps a Liouville integrable system into a Liouville integrable system and from its proof it also follows that Stäckel transform also preserves superintegrability [3]-[6].

Since $X_i = \Pi dh_i$ and $\widetilde{X}_i = \Pi d\widetilde{h}_i$ we obtain from (12)-(13) that the Hamiltonian vector fields $X_i = \Pi dh_i$ and $\widetilde{X}_i = \Pi d\widetilde{h}_i$ are related by the following transformation

$$X_{s_i} = -\sum_{j=1}^{k} \frac{\partial h_{s_i}}{\partial \alpha_j} \widetilde{X}_{s_j}, \quad i = 1, \ldots, k \tag{15}$$

$$X_i = \widetilde{X}_i - \sum_{j=1}^{k} \frac{\partial h_i}{\partial \alpha_j} \widetilde{X}_{s_j}, \quad i = 1, \ldots, n, \quad i \notin \{s_1, \ldots, s_k\} \tag{16}$$

This means that the hamiltonian vector fields X_i and $\widetilde{X}_i$ span the same n-dimensional distribution on M and also that the vector fields X_{s_i} and $\widetilde{X}_{s_i}$ span the same k-dimensional subdistribution of the above distribution. The transformation (15)-(16) can be written in matrix form as

$$X = A\widetilde{X} \tag{17}$$

where we denote $X = (X_1, \ldots, X_n)^T$ and $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_n)^T$ and where the $n \times n$ matrix A is given above.

All the vector fields X_i and $\widetilde{X}_i$ are naturally tangent to the common level surface $M_{\alpha,\widetilde{\alpha}}$ of all the Hamiltonians h_{s_i}:

$$M_{\alpha,\widetilde{\alpha}} = \{x \in M : h_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \widetilde{\alpha}_i, \, i = 1, \ldots k\}$$

(since they are obviously tangent to common level surfaces of all n Hamiltonians h_i) so that if $x_0 \in M_{\alpha,\widetilde{\alpha}}$ then the multiparameter (simultaneous) solution

$$x = x(t_1, \ldots, t_n, x_0) \tag{18}$$

of all equations in (8) starting at x_0 for $t = 0$, will always remain in $M_{\alpha,\widetilde{\alpha}}$ and the same is also true for multiparameter solutions of (9). Note that the surface $M_{\alpha,\widetilde{\alpha}}$ depends on the choice of $2k$ parameters α_i and $\widetilde{\alpha}_i$ and that its codimension is k so that $\dim M_{\alpha,\widetilde{\alpha}} = 2n - k \geq n$. Note also that, by virtue of the definition of the Stäckel transform, $M_{\alpha,\widetilde{\alpha}} = M_{\widetilde{\alpha},\alpha}$ where

$$M_{\widetilde{\alpha},\alpha} = \left\{x \in M : \widetilde{h}_{s_i}(x, \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_k) = \alpha_i, \, i = 1, \ldots k\right\}$$

The relations (15)-(16) can be reformulated in the dual language, that of reciprocal (multi-time) transformations.

Theorem 4 *The reciprocal transformation $\widetilde{t}_i = \widetilde{t}_i(t_1, \ldots, t_n, x)$, $i = 1, \ldots, n$ given by*

$$d\widetilde{t} = A^T dt \tag{19}$$

(where $dt = (dt_1, \ldots, dt_n)^T$ and $d\widetilde{t} = (d\widetilde{t}_1, \ldots, d\widetilde{t}_n)^T$) transforms the n-parameter solutions (18) of the system (8) to the n-parameter solutions $\widetilde{x} = \widetilde{x}(\widetilde{t}_1, \ldots, \widetilde{t}_n, x_0)$ of the system (9) (with the same initial condition $x(0) = x_0$) in the sense that for any $x_0 \in M$ we have

$$\widetilde{x}(\widetilde{t}_1(t_1, \ldots, t_n, x_0), \ldots, \widetilde{t}_n(t_1, \ldots, t_n, x_0), x_0) = x(t_1, \ldots, t_n, x_0)$$

for all values of t_i sufficiently close to zero.

The transformation (19) is well defined since the right hand side of (19) is an exact differential, as it follows from the above construction. It means that it is possible (at least locally) to integrate (19) and obtain an explicit transformation $\tilde{t}_i = \tilde{t}_i(t_1, \ldots, t_n, x)$ that takes multi-time (simultaneous) solutions of all hamiltonian systems (8) to multi-time solutions of all the systems in (9).

In a specific but important for us case when $k = n$ (i.e. when the number of parameters and the number of hamiltonians coincide so that the Stäckel transform consist only of the first part i.e. (4)), the matrix A simplifies insofar as

$$A_{ij} = -\frac{\partial h_i}{\partial \alpha_j}, \ \ i, j = 1, \ldots, n$$

so that the formulas (15)-(16) simplify to the single formula

$$X_i = -\sum_{j=1}^{n} \frac{\partial h_i}{\partial \alpha_j} \widetilde{X}_j, \quad i = 1, \ldots, n,$$

while (19) can be explicitly written as

$$d\tilde{t}_i = -\sum_{j=1}^{n} \frac{\partial h_j}{\partial \alpha_i} dt_j, \quad i = 1, \ldots, n. \tag{20}$$

and our manifolds $M_{\alpha,\widetilde{\alpha}}$ become in this case level surfaces for all the hamiltonians $h_i(x, \alpha)$ and also level surfaces for all the hamiltonians $\widetilde{h}_i(x, \widetilde{\alpha})$.

4 Classical Stäckel systems

Consider a set of Darboux coordinates $(\lambda, \mu) = (\lambda_1 \ldots, \lambda_n, \mu_1, \ldots, \mu_1)$ on our $2n$-dimensional Poisson manifold M equipped with a Poisson operator Π (so that $\Pi = \sum_{i<j} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}$). A classical Stäckel system is a system of n Hamiltonians H_i on M (that originally do not depend on any additional parameters α so that they can not be a subject of any Stäckel transformation) originating from a set of n separation relations [13] of the form:

$$\sigma(\lambda_i) + \sum_{j=1}^{n} H_j \lambda_i^{\gamma_j} = f(\lambda_i)\mu_i^2, \quad i = 1, \ldots, n, \tag{21}$$

where f and σ are arbitrary functions of one argument and where all $\gamma_i \in \mathbf{Z}$, $i = 1, \ldots, n$, and are such that no two γ_i coincide. Thus, a particular Stäckel system is defined by the choice of integers $\gamma_1, \ldots, \gamma_n$ and by the choice of functions f and σ. Customary one can also treat this system of relations as n points on (n copies of) the following *separation curve*

$$P(\lambda, H) \equiv \sigma(\lambda) + \sum_{j=1}^{n} H_j \lambda^{\gamma_j} = f(\lambda)\mu^2, \tag{22}$$

in $\lambda\mu$ plane which helps us to avoid writing too many indices. The relations (21) (or n copies of (22)) constitute a system of n equations linear in the unknowns H_i. Solving these relations with respect to H_i we obtain n commuting (since the right-hand sides of formulas (21) commute) with respect to Π Hamiltonians (known in literature as Stäckel Hamiltonians) on M of the form

$$H_i = \mu^T K_i G \mu + V_i(\lambda) \ \ i = 1, \ldots, n, \tag{23}$$

where we denote $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and $\mu = (\mu_1, \ldots, \mu_n)^T$. The functions H_i can be interpreted as n quadratic in momenta μ hamiltonians on the phase space $M = T^*\mathcal{Q}$ cotangent to a Riemannian manifold $\mathcal{Q}$ (so that $\lambda_1, \ldots, \lambda_n$ are coordinates on $\mathcal{Q}$) equipped with the contravariant metric tensor G depending on function f and the choice of the set γ. The objects K_i in (23) can be interpreted as $(1,1)$-type Killing tensors on $\mathcal{Q}$ for the metric G. The metric tensor G and all the Killing tensors K_i are diagonal in λ-variables. Note that by the very construction of H_i the variables (λ, μ) are separation variables for all the hamiltonians in (23) in the sense that the Hamilton-Jacobi equations associated with all $H_i = a_i$ admit additively separable solutions $W = \sum_{i=1}^{n} W_i(\lambda_i, a)$.

The relations (21) can be written in a matrix form as

$$S_\gamma H = U$$

where $H=(H_1,\ldots,H_n)^T$, and where U is a Stäckel vector of the form

$$U=(f(\lambda_1)\mu_1^2-\sigma(\lambda_1),\ldots,f(\lambda_n)\mu_n^2-\sigma(\lambda_n))^T, \tag{24}$$

while the matrix S_γ is a classical Stäckel matrix of the form

$$S_\gamma=\begin{pmatrix} \lambda_1^{\gamma_1} & \cdots & \lambda_1^{\gamma_n} \\ \vdots & \ddots & \vdots \\ \lambda_n^{\gamma_1} & \cdots & \lambda_n^{\gamma_n} \end{pmatrix}. \tag{25}$$

Note that our assumption that no γ_i coincide means that $\det(S_\gamma)\neq 0$. Thus, the hamiltonians (23) can be obtained in a matrix form as

$$H=S_\gamma^{-1}U,$$

which also means that the metric G in (23) can be expressed as

$$G=\mathrm{diag}\left(f(\lambda_1)\left(S_\gamma^{-1}\right)_{11},\ldots,f(\lambda_n)\left(S_\gamma^{-1}\right)_{1n}\right),$$

so that the Killing tensors K_i in (23) are

$$K_i=\mathrm{diag}\left(\left(S_\gamma^{-1}\right)_{i1}/\left(S_\gamma^{-1}\right)_{11},\ldots,\left(S_\gamma^{-1}\right)_{in}/\left(S_\gamma^{-1}\right)_{1n}\right),\quad i=1,\ldots,n$$

(note that $K_1=I$). Let us now turn our attention to the scalar functions $V_i:\mathcal{Q}\to\mathbf{R}$ in (23). Relations (22) and (23) imply that $V_i(\lambda)$ satisfy the following separation curve

$$\sigma(\lambda)+V_1\lambda^{\gamma_1}+V_2\lambda^{\gamma_2}+\ldots+V_n\lambda^{\gamma_n}=0, \tag{26}$$

so that they depend on the choice of integers γ_i and the choice of the function σ. We will therefore denote them as $V_i^{(\sigma)}$. In case when $\sigma(\lambda)$ is a monomial, i.e. when $\sigma(\lambda)=\lambda^k$ with $k\in\mathbf{Z}$, V_i depend on k and they will be denoted by $V_i^{(k)}$ (so that $V_i^{(k)}=V_i^{(\lambda^k)}$) to shorten the notation. Thus, the potentials $V_i^{(k)}(\lambda)$ (they still depend on all γ_i) satisfy the following separation curve

$$\lambda^k+V_1^{(k)}\lambda^{\gamma_1}+V_2^{(k)}\lambda^{\gamma_2}+\ldots+V_n^{(k)}\lambda^{\gamma_n}=0, \tag{27}$$

which in matrix form can be written as

$$S_\gamma V^{(k)}=-\Lambda^k(1,\ldots,1)^T, \tag{28}$$

where $V^{(k)}=(V_1^{(k)},\ldots,V_n^{(k)})^T$ and $\Lambda=$diag$(\lambda_1,\ldots,\lambda_n)$. This means that

$$V^{(0)}=-S_\gamma^{-1}(1,\ldots,1)^T$$

so that $V^{(1)}=S_\gamma^{-1}\Lambda S_\gamma V^{(0)}$, $V^{(2)}=S_\gamma^{-1}\Lambda^2 S_\gamma V^{(0)}=(S_\gamma^{-1}\Lambda S_\gamma)(S_\gamma^{-1}\Lambda S_\gamma)V^{(0)}$ and so on. Similar argument applies also for negative k. Thus, denoting

$$F_\gamma=S_\gamma^{-1}\Lambda S_\gamma \tag{29}$$

we get the compact formula for the potentials $V^{(k)}$ (presented first in [12]):

$$V^{(k)}=F_\gamma^k V^{(0)},\quad k\in\mathbf{Z}. \tag{30}$$

It is now an immediate consequence of the above formulas that for any meromorphic function $\sigma(\lambda)$ we have

$$V^{(\sigma)}=\sigma(F_\gamma)V^{(0)}, \tag{31}$$

where $V^{(\sigma)}=(V_1^{(\sigma)},\ldots,V_n^{(\sigma)})$. The matrix F_γ given in (29) has been called control matrix in [14] where it appeared in the context of quasi-bi-Hamiltonian representation. Notice also that if the system (22) is normed by $\gamma_n=1$, then the potential $V^{(0)}$ attains a particularly simple form $V^{(0)}=(0,\ldots,0,-1)^T$. This follows immediately from (28).

Now, by writing down the inverse Jacobi problem for all the Hamiltonians (23) we can arrive at the following remark that will be useful in the next section when we discuss reciprocal transforms between different Stäckel systems.

Remark 5 *On the level surface $M_a = \{x \in M : H_i = a_i \in \mathbf{R}\}$ the mutliparameter (multi-time) solutions $\lambda_i = \lambda_i(t_1, \ldots, t_n, x_0)$ of all Hamiltonian systems defined by the separation curve (22) or equivalently by all Hamiltonians (23) attain the following Abel-Jacobi differential form*

$$dt = S_\gamma^T \frac{d\lambda}{\sqrt{f(\lambda)P(\lambda, a)}}, \tag{32}$$

where $d\lambda/\sqrt{f(\lambda)P(\lambda,a)}$ means a column vector with components $d\lambda_i/\sqrt{f(\lambda_i)P(\lambda_i,a)}$.

Note that solutions (32) define in a standard (canonical) way the corresponding multi-time solutions for the momenta $\mu_i = \mu_i(t_1, \ldots, t_n, x_0)$.

A particular subclass of Stäckel systems is given by choosing $\gamma_i = n - i$. The separation curve (22) attains then the form

$$\sigma(\lambda) + \sum_{j=1}^{n} H_j \lambda^{n-j} = f(\lambda)\mu^2$$

and the originating hamiltonians H_i constitute a completely integrable system that we call a system of Benenti type due to S. Benenti's contribution to the study of these objects [15],[16]. In this case it is possible to give compact formulas for many objects introduced above. Thus, for example, the metric G and the Killing tensors K_i in (23) are given explicitly as

$$G = \operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1}, \ldots, \frac{f(\lambda_n)}{\Delta_n}\right), \quad \Delta_i = \prod_{j \neq i}(\lambda_i - \lambda_j)$$

$$K_i = -\operatorname{diag}\left(\frac{\partial \rho_i}{\partial \lambda_1}, \cdots, \frac{\partial \rho_i}{\partial \lambda_n}\right) \qquad i = 1, \ldots, n.$$

Here and below $\rho_i = \rho_i(\lambda)$ are Viète polynomials (signed symmetric polynomials) in λ:

$$\rho_i(\lambda) = (-1)^i \sum_{1 \le s_1 < s_2 < \ldots < s_i \le n} \lambda_{s_1} \ldots \lambda_{s_i}, \quad i = 1, \ldots, n \tag{33}$$

that can also be considered as new coordinates on the Riemannian manifold $\mathcal{Q}$ (we will then refer to them as Viète coordinates). Notice again that the Killing tensors K_i do not depend on a particular choice of f and σ. It can be shown that as long as f is a polynomial of degree $\le n$ then the metric G is flat while if f is a polynomial of degree $n+1$ then G has constant but non-zero curvature. For the Benenti class the control matrix F_γ in (29) attains the simple form

$$F = \begin{pmatrix} -\rho_1 & 1 & & \\ -\rho_2 & & \ddots & \\ \vdots & & & 1 \\ -\rho_n & 0 & \cdots & 0 \end{pmatrix} \tag{34}$$

and since $V^{(0)} = (0, 0, \ldots, 0, -1)^T$ we easily obtain that the potentials $V^{(1)} = FV^{(0)} = (0, 0, \ldots 0, -1, 0)^T$, $V^{(2)} = F^2 V^{(0)} = (0, 0, \ldots 0, -1, 0, 0)^T$ up to $V^{(n-1)} = F^{n-1}V^{(0)} = (-1, 0, \ldots, 0)^T$, are trivial (constant), $V^{(n)} = F^n V^{(0)} = (\rho_1, \ldots, \rho_n)$ is the first nontrivial positive potential while $V^{(-1)} = F^{-1}V^{(0)}$ $= (1/\rho_n, \rho_1/\rho_n, \ldots, \rho_{n-1}/\rho_n)^T$ and so on. More information on Benenti systems can be found in [17, 18, 19].

5 Stäckel equivalence of Stäckel systems

We will now turn to the main question of this article: how to relate two Stäckel systems by a single Stäckel transform and in such a way that their solutions are related by a reciprocal transform? As we mentioned above, The Hamiltonians H_i defined by (21) or by (22) do not depend on any additional parameters α_i so in order to perform a Stäckel transform on (21) we have to embed it into a parameter-dependent

system. Of course, there is infinitely many ways of embedding of our Stäckel system into an n-parameter system but the choice below is natural in the sense that the corresponding Stäckel transform transforms a Stäckel system into a new Stäckel system. Thus, consider n Hamiltonians $h_i = h_i(\lambda, \mu, \alpha)$ defined by the separation curve

$$P(\lambda, h, \alpha) \equiv \sigma(\lambda) + \sum_{j=1}^{n} h_j \lambda^{\gamma_j} + R^{-1}(\lambda) \sum_{j=1}^{n} \alpha_j \lambda^{\delta_j} = f(\lambda)\mu^2 \tag{35}$$

where $\gamma_1, \ldots, \gamma_n$ and $\delta_1, \ldots, \delta_n$ are two sequences of integers such that no two γ_i coincide and similarly no two δ_i coincide (but we do admit the possibility that some or all of γ_i coincide with some δ_i) and where $R(\lambda)$ is an arbitrary meromorphic function of one variable so that

$$R(\lambda) = \prod_{s=1}^{k_1} (\lambda - \beta_s) \prod_{s=1}^{k_2} (\lambda - \beta_s')^{-1}$$

for some (complex in general) constants $\beta_1, \ldots, \beta_{k_1}$ and $\beta_1', \ldots, \beta_{k_2}'$. This function can be generalized to a matrix function i.e. we define, for any $n \times n$ matrix A (λ-dependent or not)

$$R(A) = \prod_{s=1}^{k_1} (A - \beta_s) \prod_{s=1}^{k_2} (A - \beta_s')^{-1} \tag{36}$$

(note that all the terms in (36) commute so that there is no ordering problem here). The relations (35) can now be written in a matrix form as

$$S_\gamma h + R^{-1}(\Lambda) S_\delta \alpha = U \tag{37}$$

where S_γ and S_δ are two Stäckel matrices given by (25) (so that $(S_\gamma)_{ij} = \lambda_i^{\gamma_j}$ and $(S_\delta)_{ij} = \lambda_i^{\delta_j}$), $h = (h_1, \ldots, h_n)^T$ is the column vector consisting of Hamiltonians h_i, $\alpha = (\alpha_1, \ldots, \alpha_n)^T$, U is the column vector given in (24) and where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ as before. Solving (37) with respect to h we obtain

$$h = S_\gamma^{-1} U - S_\gamma^{-1} R^{-1}(\Lambda) S_\delta \alpha = S_\gamma^{-1} U - S_\gamma^{-1} R^{-1}(\Lambda) S_\gamma S_\gamma^{-1} S_\delta \alpha. \tag{38}$$

Lemma 6 *In the notation as above*

$$S_\gamma^{-1} R(\Lambda) S_\gamma = R(F_\gamma).$$

Proof. We show it for $R(\lambda) = \lambda - \beta$ as the general statement follows easily by developing the argument below.

$$S_\gamma^{-1} (\Lambda - \beta) S_\gamma = S_\gamma^{-1} \Lambda S_\gamma - \beta I = F_\gamma - \beta I = R(F_\gamma).$$

■

Thus, introducing the shorthand notation

$$W_{\delta,\gamma} = -S_\gamma^{-1} S_\delta$$

we see that (38) can be written as

$$h = H + R^{-1}(F_\gamma) W_{\delta,\gamma} \alpha \tag{39}$$

where $H = S_\gamma^{-1} U$ is the part of h that is independent of parameters α_i (cf. (6)). Let us shortly analyze the structure of the matrix $W_{\delta,\gamma} = -S_\gamma^{-1} S_\delta$. Assume that $\sigma(\lambda)$ in (26) is a polynomial of the form $\sigma(\lambda) = \sum_{i=1}^{n} \xi_i \lambda^{\delta_i}$. Then, as it follows from (26) and from the definition of potentials $V^{(k)}$:

$$V^{(\sigma)} = \sum_{i=1}^{n} \xi_i V^{(\delta_i)}.$$

On the other hand, the formula (26) can now be written as

$$S_\gamma V^{(\sigma)} + S_\delta \xi = 0$$

so that $V^{(\sigma)} = -S_\gamma^{-1} S_\delta \xi = W_{\delta,\gamma}\xi$ which implies that

$$(W_{\delta,\gamma})_{ij} = V_i^{(\delta_j)},$$

where $V^{(\delta_j)} = F_\gamma^{\delta_j} V^{(0)}$ in accordance with (30). Therefore, the formula (39) can be written as

$$h_i = H_i + \sum_{j,k=1}^{n} \left(R^{-1}(F_\gamma)\right)_{ij} V_j^{(\delta_k)} \alpha_k, \quad i = 1, \ldots, n. \tag{40}$$

Let us now perform an n-parameter Stäckel transform of the system given by the curve (35). Since the number of parameters α_i and the number of Hamiltonians h_i are both the same ($= n$) the Stäckel transform consists only of part (4) and is therefore generated by the relation $h = \widetilde{\alpha}$ (which implies $\widetilde{h} = \alpha$) in the vector notation as above. We are now in position to formulate the main theorem of this paper.

Theorem 7 *The n-parameter Stäckel transform generated by $h = \widetilde{\alpha}$ transforms the set of n Hamiltonians h defined by (35) into the following set of Hamiltonians*

$$\widetilde{h} = -W_{\delta,\gamma}^{-1} R(F_\gamma) H + W_{\delta,\gamma}^{-1} R(F_\gamma)\widetilde{\alpha} \tag{41}$$

(where $\widetilde{h} = (\widetilde{h}_1, \ldots, \widetilde{h}_n)^T$ and similarly $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)^T$) which constitute a new Stäckel system with the separation curve of the form

$$\widetilde{P}(\lambda, \widetilde{h}, \widetilde{\alpha}) \equiv R(\lambda)\sigma(\lambda) + R(\lambda)\sum_{j=1}^{n} \widetilde{\alpha}_j \lambda^{\gamma_j} + \sum_{j=1}^{n} \widetilde{h}_j \lambda^{\delta_j} = R(\lambda) f(\lambda)\mu^2. \tag{42}$$

Moreover, the reciprocal transformation

$$d\widetilde{t} = -R^{-1}(F_\delta^T) W_{\delta,\gamma}^T dt = -W_{\delta,\gamma}^T R^{-1}(F_\gamma^T) dt \tag{43}$$

transforms n-time solutions $x = x(t_1, \ldots, t_n, x_0)$ of the system (35) into n-time solutions $\widetilde{x} = \widetilde{x}(\widetilde{t}_1, \ldots, \widetilde{t}_n, x_0)$ of the system (42).

Note that in spite of the fact that we introduced both systems in the (λ, μ)-variables the matrix formulas (41) and (43) are not tensor and that they are coordinate-free. They can be therefore freely applied in any coordinate system on M, which will be used in the examples further on.
Proof. Multiplying the curve (35) by $R(\lambda)$ we obtain

$$R(\lambda)\sigma(\lambda) + R(\lambda)\sum_{j=1}^{n} h_j \lambda^{\gamma_j} + \sum_{j=1}^{n} \alpha_j \lambda^{\delta_j} = R(\lambda) f(\lambda)\mu^2$$

which after the Stäckel transform $h = \widetilde{\alpha}$ (so that $\widetilde{h} = \alpha$) obviously attains the form (42). Let us therefore show the formula (41). The separation relations implied by (42) can be written in matrix form as

$$R(\Lambda) S_\gamma \widetilde{\alpha} + S_\delta \widetilde{h} = R(\Lambda) U$$

with the column vector U as in (24). Solving this with respect to $\widetilde{h}$ we obtain

$$\begin{aligned}
\widetilde{h} &= S_\delta^{-1} R(\Lambda) U - S_\delta^{-1} R(\Lambda) S_\gamma \widetilde{\alpha} = S_\delta^{-1} R(\Lambda) S_\gamma H - S_\delta^{-1} R(\Lambda) S_\gamma \widetilde{\alpha} \\
&= \left(S_\delta^{-1} R(\Lambda) S_\delta\right) \left(S_\gamma^{-1} S_\delta\right)^{-1} H - \left(S_\delta^{-1} R(\Lambda) S_\delta\right) \left(S_\gamma^{-1} S_\delta\right)^{-1} \widetilde{\alpha} \\
&= -R(F_\delta) W_{\delta,\gamma}^{-1} H - R(F_\delta) W_{\delta,\gamma}^{-1} \widetilde{\alpha}
\end{aligned}$$

so the only remaining thing is to show that $R(F_\delta) W_{\delta,\gamma}^{-1} = W_{\delta,\gamma}^{-1} R(F_\gamma)$ which is equivalent to the statement

$$W_{\delta,\gamma} R(F_\delta) = R(F_\gamma) W_{\delta,\gamma}$$

that can easily be proved in a fashion similar to proof of Lemma 6. Finally, the formula (43) is obtained by inserting (39) into (20)

$$d\widetilde{t} = -\left(\frac{\partial h}{\partial \alpha}\right)^T dt = -\left(R^{-1}(F_\gamma)W_{\delta,\gamma}\right)^T dt,$$

where we use the fact that $R(A)^T = R(A^T)$. ■

Let us also remark that the relations (41) can be explicitly written as (cf. (40)):

$$\widetilde{h}_i = -\sum_{j,k=1}^{n} R(F_\delta)_{ij}\widetilde{V_j}^{(\delta_k)} H_k + \sum_{j,k=1}^{n} R(F_\delta)_{ij}\widetilde{V_j}^{(\delta_k)}\widetilde{\alpha}_k, \quad i = 1,\ldots,n$$

where the potentials $\widetilde{V_j}^{(k)}$ are defined by the separation curve (cf. (27))

$$\lambda^k + \widetilde{V}_1^{(k)}\lambda^{\delta_1} + \widetilde{V}_2^{(k)}\lambda^{\delta_2} + \ldots + \widetilde{V}_n^{(k)}\lambda^{\delta_1} = 0 \tag{44}$$

so that

$$\widetilde{V}^{(k)} = F_\delta^k V^{(0)}.$$

Notice that due to the form of (27) and (44) the potentials $V^{(k)}$ and $\widetilde{V}^{(k)}$ are related by the Stäckel transform $V^{(k)} = \widetilde{\alpha}, \widetilde{V}^{(k)} = \alpha$.

Let us also present an alternative way of proving the formula (43), directly involving solutions of (35) and (42). It follows from Remark 5 and from the above considerations that the multi-time solutions of the systems (35) and (42) on any common level surface $M_{\alpha,\widetilde{\alpha}}$ attain the form

$$dt = S_\gamma^T \frac{d\lambda}{\sqrt{f(\lambda)P(\lambda,\widetilde{\alpha},\alpha)}}, \qquad d\widetilde{t} = S_\delta^T \frac{d\lambda}{\sqrt{R(\lambda)f(\lambda)\widetilde{P}(\lambda,\alpha,\widetilde{\alpha})}}. \tag{45}$$

Now, it is easy to see that $\widetilde{P}(\lambda,\alpha,\widetilde{\alpha}) = R(\lambda)P(\lambda,\widetilde{\alpha},\alpha)$ so that, by (45) and by the fact that $R(\Lambda)$ is symmetric

$$d\widetilde{t} = S_\delta^T \frac{d\lambda}{\sqrt{R(\lambda)f(\lambda)\widetilde{P}(\lambda,\alpha,\widetilde{\alpha})}} = S_\delta^T \frac{d\lambda}{\sqrt{R^2(\lambda)f(\lambda)P(\lambda,\widetilde{\alpha},\alpha)}} = S_\delta^T R^{-1}(\Lambda)\frac{d\lambda}{\sqrt{f(\lambda)P(\lambda,\widetilde{\alpha},\alpha)}} =$$
$$= S_\delta^T R^{-1}(\Lambda)\left(S_\gamma^T\right)^{-1} dt = \left(S_\gamma^{-1}R^{-1}(\Lambda)S_\delta\right)^T dt = \left(S_\gamma^{-1}S_\delta S_\delta^{-1}R^{-1}(\Lambda)S_\delta\right)^T dt = -(W_{\delta,\gamma}R^{-1}(F_\delta))^T dt$$

thus yielding

$$d\widetilde{t} = -R^{-1}(F_\delta^T)W_{\delta,\gamma}^T dt = -W_{\delta,\gamma}^T R^{-1}(F_\gamma^T)dt,$$

which is what we wanted to prove.

On the level of Stäckel transforms Theorem 7 leads to the following corollary:

Corollary 8 *Assume that $f_1(\lambda) \neq 0$ and $f_2(\lambda) \neq 0$. Any two Stäckel systems of the form*

$$\sigma_1(\lambda) + \sum_{j=1}^{n} H_j\lambda^{\gamma_j} = f_1(\lambda)\mu^2$$
$$\sigma_2(\lambda) + \sum_{j=1}^{n} \widetilde{H}_j\lambda^{\delta_j} = f_2(\lambda)\mu^2$$

that satisfy the condition

$$f_2(\lambda)\sigma_1(\lambda) = \sigma_2(\lambda)f_1(\lambda) \tag{46}$$

are Stäckel-related by the single Stäckel transform

$$\widetilde{H} = -W_{\delta,\gamma}^{-1}R(F_\gamma)H \tag{47}$$

with $R(\lambda) = \frac{f_2(\lambda)}{f_1(\lambda)}$. In particular, any two geodesic Stäckel systems (i.e. with $\sigma_1 = \sigma_2 = 0$) are connected by the Stäckel transform (47) with $R(\lambda) = \frac{f_2(\lambda)}{f_1(\lambda)}$.

Proof. By Theorem 7, the Stäckel transform (47) with $R(\lambda)=\frac{f_2(\lambda)}{f_1(\lambda)}$ transforms the first of the above Stäckel systems into the second one provided that $\frac{f_2(\lambda)}{f_1(\lambda)}=\frac{g_2(\lambda)}{g_l(\lambda)}$ which is exactly the condition (46). In case of geodesic systems, the condition (46) is always satisfied. ■

Proposition 9 *The condition (46) splits all Stäckel systems of the form (21) into equivalence classes since it is an equivalence relation.*

Proof. Indeed, if $\frac{f_2(\lambda)}{f_1(\lambda)}=\frac{g_2(\lambda)}{g_1(\lambda)}=R_1(\lambda)$ and $\frac{f_3(\lambda)}{f_2(\lambda)}=\frac{g_3(\lambda)}{g_2(\lambda)}=R_2(\lambda)$ then $\frac{f_3(\lambda)}{f_1(\lambda)}=\frac{g_3(\lambda)}{g_1(\lambda)}=R_2(\lambda)R_1(\lambda)$ so this relation is transitive. Further, if $\frac{f_2(\lambda)}{f_1(\lambda)}=\frac{g_2(\lambda)}{g_1(\lambda)}=R(\lambda)$ then $\frac{f_1(\lambda)}{f_2(\lambda)}=\frac{g_1(\lambda)}{g_2(\lambda)}=\frac{1}{R(\lambda)}$ so this relation is reflexive. Finally, $\frac{f_1(\lambda)}{f_1(\lambda)}=\frac{g_1(\lambda)}{g_1(\lambda)}=1$ so it is a symmetric relation.
■

Our formulas contain two special cases: when $\gamma=\delta$ and when $R=1$. In the first case (i.e. when $\gamma_i=\delta_i$, $i=1,\ldots,n$; Benenti systems are in this class) we relate systems belonging to the same class, where the class is understanding as a fixed sequence $\gamma_1,...,\gamma_n$, and differ by f and σ.The matrix $W_{\gamma,\gamma}=-I$ while $F_\delta=F_\gamma$ so that the formula (41) becomes

$$\widetilde{h}=R(F_\gamma)H-R(F_\gamma)\widetilde{\alpha} \tag{48}$$

while (43) attains the form

$$d\widetilde{t}=R^{-1}(F_\gamma^T)dt.$$

In the second case ($R=1$) we relate systems from different classes, i.e. $(\gamma_1,...,\gamma_n)$ and $(\delta_1,...,\delta_n)$ respectively, which share the same f and σ. The formula (41) becomes

$$\widetilde{h}=-W_{\delta,\gamma}^{-1}H+W_{\delta,\gamma}^{-1}\widetilde{\alpha}$$

while the formula (43) attains the form

$$d\widetilde{t}=-W_{\delta,\gamma}^{T}dt. \tag{49}$$

Thus, the general transformation between the systems (35) and (42) can be considered as composition of two transformations: a map between two Stäckel systems from the same class (i.e. with $\gamma=\delta$) but with different f (i.e. metrics) and/or different σ and the transformation between two Stäckel systems sharing the same f and σ but from different classes. Both these transformations commute.

6 Examples

We will now present two examples of our formulas. In order to relate to known integrable systems we will present both examples in their natural (physical) coordinates.

In our fist example we will relate two families of separation curves for $n=2$, namely

$$P(\lambda,h,\alpha)\equiv\sigma(\lambda)+h_1\lambda+h_2+\lambda(\alpha_1\lambda^2+\alpha_2)=\frac{1}{2}\lambda\mu^2 \tag{50}$$

and

$$\widetilde{P}(\lambda,\widetilde{h},\widetilde{\alpha})\equiv\sigma(\lambda)\lambda^{-1}+\lambda^{-1}(\widetilde{\alpha}_1\lambda+\widetilde{\alpha}_2)+\widetilde{h}_1\lambda^2+\widetilde{h}_2=\frac{1}{2}\mu^2 \tag{51}$$

which are particular cases of (35) respectively (42) with $(\gamma_1,\gamma_2)=(1,0)$, $(\delta_1,\delta_2)=(2,0)$ and with $R=\lambda^{-1}$, while $\sigma(\lambda)$ is for now assumed to be an arbitrary rational function of λ. The family (50) contains in particular a well known Henon-Hèiles (HH) system while the family (51) contains in particular Drach system [20]. Note that in both of the above curves one γ_i coincides with one δ_i and therefore we can regroup the terms in the above curves to obtain

$$\sigma(\lambda)+\alpha_1\lambda^3+(h_1+\alpha_2)\lambda+h_2=\frac{1}{2}\lambda\mu^2 \tag{52}$$

for the HH family (50) and

$$\sigma(\lambda)\lambda^{-1}+\widetilde{h}_1\lambda^2+(\widetilde{h}_2+\widetilde{\alpha}_1)+\widetilde{\alpha}_2\lambda^{-1}=\frac{1}{2}\mu^2 \tag{53}$$

for the Drach family (51). Since both metrics are flat we will now construct both systems in their respective flat coordinates. As an intermediate step, we will write the systems in Viète coordinates (33) that now attain the form

$$\rho_1 = -\lambda_1 - \lambda_2, \quad \rho_2 = \lambda_1\lambda_2. \tag{54}$$

In the above coordinates the control matrix F_γ (cf. (29) and (34)) of the HH family is

$$F_\gamma = \begin{pmatrix} -\rho_1 & 1 \\ -\rho_2 & 0 \end{pmatrix}$$

and as we remember it is coordinate free. The passage to the flat coordinates (x_1, x_2) is given by the point transformation [21]

$$\rho_1 = x_1, \quad \rho_2 = -\frac{1}{4}x_2^2.$$

The metric G and the Killing tensor K_2 are now

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & -\frac{1}{2}x_2 \\ -\frac{1}{2}x_2 & x_1 \end{pmatrix}$$

so that geodesic (i.e. with $\sigma(\lambda) = 0$) Hamiltonians of (52) are

$$E_1 = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2, \qquad E_2 = \frac{1}{2}x_1 y_2^2 - \frac{1}{2}x_2 y_1 y_2,$$

where $(y_1, y_2)^T$ are momenta conjugate to (x_1, x_2). Note that (x_1, x_2) are not only flat but also orthogonal coordinates for (52). The matrix F_γ is now

$$F_\gamma = \begin{pmatrix} -x_1 & 1 \\ \frac{1}{4}x_2^2 & 0 \end{pmatrix}$$

so that the Hamiltonians of (52) in the flat coordinates are

$$\begin{aligned} h_1 &= E_1 + V_1^{(\sigma)} - \left(x_1^2 + \frac{1}{4}x_2^2\right)\alpha_1 - \alpha_2 \\ h_1 &= E_1 + V_2^{(\sigma)} + \frac{1}{4}x_1 x_2^2 \alpha_1, \end{aligned}$$

where due to (31) we have $V^{(\sigma)} = \sigma(F_\gamma)V^{(0)} = \sigma(F_\gamma)(0, -1)^T$. Let us now confine ourselves to the following three-parameter set of separable potentials:

$$\sigma(\lambda) = b_1\lambda^5 + b_2\lambda^4 + b_3\lambda^2, \quad b_i \in \mathbf{R}. \tag{55}$$

We can now calculate $V^{(\sigma)}$ explicitly and we find

$$\begin{aligned} h_1 &= E_1 - b_1\left(x_1^4 + \frac{3}{4}x_1^2x_2^2 + \frac{1}{16}x_2^4\right) + b_2\left(x_1^3 + \frac{1}{2}x_1x_2^2\right) + b_3x_1 - \left(x_1^2 + \frac{1}{4}x_2^2\right)\alpha_1 - \alpha_2 \\ h_2 &= E_2 + b_1\left(\frac{1}{4}x_1^3x_2^2 + \frac{1}{8}x_1x_2^4\right) - b_2\left(\frac{1}{4}x_1^2x_2^2 + \frac{1}{16}x_2^3\right) - \frac{1}{4}b_3x_2^2 + \frac{1}{4}x_1x_2^2\alpha_1 \end{aligned}$$

and for $\alpha_1 = \alpha_2 = 0$, $b_1 = b_3 = 0, b_2 = 1$ we receive the classical Hènon-Heiles system.

Consider now the Drach family (53). The matrix F_δ in Viète coordinates (54) has the form

$$F_\delta = \begin{pmatrix} \frac{-\rho_1^2+\rho_2}{\rho_1} & -\frac{1}{\rho_1} \\ \frac{\rho_2^2}{\rho_1} & -\frac{\rho_2}{\rho_1} \end{pmatrix}.$$

We pass now to flat but non-orthogonal coordinates $(x, y\,)$ given by [20]

$$\rho_1 = -2x^{\frac{1}{2}}, \quad \rho_2 = x - y.$$

The metric $\widetilde{G}$ and the Killing tensor $\widetilde{K}_2$ of this system are

$$\widetilde{G} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \widetilde{K}_2 = \begin{pmatrix} -(x+y) & 2x \\ 2y & -(x+y) \end{pmatrix}$$

so that the geodesic Hamiltonians of (53) are

$$\widetilde{E}_1 = \frac{1}{2}p_x p_y, \qquad \widetilde{E}_2 = \frac{1}{2}xp_x^2 + \frac{1}{2}yp_y^2 - \frac{1}{2}(x+y)p_x p_y,$$

where (p_x, p_y) are momenta conjugate to (x, y). The matrix F_δ becomes

$$F_\delta = \begin{pmatrix} \frac{1}{2}x^{-\frac{1}{2}}(3x+y) & \frac{1}{2}x^{-\frac{1}{2}} \\ -\frac{1}{2}x^{-\frac{1}{2}}(x-y)^2 & \frac{1}{2}x^{-\frac{1}{2}}(x-y) \end{pmatrix}$$

and hence

$$\begin{aligned} \widetilde{h}_1 &= \widetilde{E}_1 + \widetilde{V}_1^{(\sigma)} + \frac{1}{2}x^{-\frac{1}{2}}(x-y)^{-1}\widetilde{\alpha}_2 \\ \widetilde{h}_2 &= \widetilde{E}_2 + \widetilde{V}_2^{(\sigma)} - \widetilde{\alpha}_1 - \frac{1}{2}x^{-\frac{1}{2}}(x-y)^{-1}(3x+y)\widetilde{\alpha}_2 \end{aligned}$$

where due to (31) and to (53) we have $\widetilde{V}^{(\sigma)} = \sigma(F_\delta)F_\delta^{-1}\widetilde{V}^{(0)} = \sigma(F_\delta)F_\delta^{-1}(0,-1)^T$. For our particular choice of $\sigma(\lambda)$ as in (55) we have $\sigma(\lambda)\lambda^{-1} = b_1\lambda^4 + b_2\lambda^3 + b_3\lambda$ and the Hamiltonians $\widetilde{h}_i$ attain the explicit form

$$\begin{aligned} \widetilde{h}_1 &= \widetilde{E}_1 - 2b_1(x+y) - \frac{1}{2}b_2x^{-\frac{1}{2}}(3x+y) - \frac{1}{2}b_3x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}(x-y)^{-1}\widetilde{\alpha}_2 \\ \widetilde{h}_2 &= \widetilde{E}_2 + b_1(x-y)^2 + \frac{1}{2}b_2x^{-\frac{1}{2}}(x-y)^2 - \frac{1}{2}b_3x^{-\frac{1}{2}}(x-y) - \widetilde{\alpha}_1 - \frac{1}{2}x^{-\frac{1}{2}}(x-y)^{-1}(3x+y)\widetilde{\alpha}_2. \end{aligned}$$

The above Hamiltonians become, after identification of constants $-2b_1 = \alpha$, $-\frac{1}{2}b_3 = \beta$, $-\frac{1}{2}b_2 = \gamma$, $\widetilde{\alpha}_2 = 0$ identical with the three-parameter Drach systems given in [20].

Let us now perform the transform between both families, i.e. between (52) and (53). According to formula (41) in Theorem 7 the parameter independent parts of Hamiltonians transform as

$$\widetilde{H} = -W_{\delta,\gamma}^{-1}R(F_\gamma)H \equiv CH$$

with $R(F_\gamma) = F_\gamma^{-1}$ and with

$$W_{\delta,\gamma} = \begin{pmatrix} V_1^{(2)} & 0 \\ V_2^{(2)} & -1 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ -\frac{1}{4}x_2^2 & -1 \end{pmatrix},$$

so that

$$C = \begin{pmatrix} 0 & -\frac{4}{x_1x_2^2} \\ 1 & \frac{4x_1^2+x_2^2}{x_1x_2^2} \end{pmatrix}.$$

Moreover, the map (43) between solutions (45) of (52) and (53) becomes

$$d\widetilde{t} = -W_{\delta,\gamma}^T R^{-1}(F_\gamma^T)dt = A^T dt$$

where

$$A = C^{-1} = -\frac{\partial h}{\partial \alpha} = \begin{pmatrix} x_1^2 + \frac{1}{4}x_2^2 & 1 \\ -\frac{1}{4}x_1x_2^2 & 0 \end{pmatrix}$$

and this map leaves the common level surface $M_{\alpha,\widetilde{\alpha}}$ invariant. Note also that now $\widetilde{P}(\lambda, \alpha_1, \alpha_2, \widetilde{\alpha}_1, \widetilde{\alpha}_2) = \lambda^{-1}P(\lambda, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \alpha_1, \alpha_2)$. The point transformation between flat coordinates of both families is

$$x_1 = -2x^{1/2}, \quad x_2 = 2(y-x)^{1/2} \implies x = \frac{1}{2}x_1^2, \quad y = \frac{1}{4}x_1^2 + \frac{1}{4}x_2^2.$$

In our second example we will relate the HH family of separable potentials with a family of elliptic separable potentials. Both systems will belong to the same class of Stäckel systems i.e. $\gamma_i = \delta_i$ for $i = 1, 2$ now (so that $W_{\gamma,\delta} = -I$ and $F_\gamma = F_\delta$ which is valid in any coordinate system) but they have different metrics. Let us thus first recollect some basic fact about generalized elliptic coordinates and a hierarchy of elliptic separable potentials [22]. Denote by $(q_1, ..., q_n)$ the Euclidian coordinates on $\mathbf{R}^n$ and by $(p_1, ..., p_n)$ the conjugate momenta. The generalized Jacobi elliptic coordinates $(\lambda_1, ..., \lambda_n)$ are defined by

$$1 + \frac{1}{4}\sum_{k=1}^{n}\frac{q_k^2}{(z-\beta_k)} = \frac{\prod_{j=1}^{n}(z-\lambda_j)}{\prod_{j=1}^{n}(z-\beta_j)}, \tag{56}$$

where β_i are nonzero different constants. Let us introduce the following abbreviations

$$B(z) = \prod_{j=1}^{n}(z-\beta_j), \quad \Lambda(z) = \prod_{j=1}^{n}(z-\lambda_j), \quad \frac{B(z)}{(z-\beta_k)} = B_k(z) = -\sum_{j=1}^{n}\frac{\partial\rho_j(\beta)}{\partial\beta_k}z^{n-j}, \tag{57}$$

where ρ_j are Viète polynomials with respect to its arguments. Then (56) takes the form

$$B(z) + \frac{1}{4}\sum_{k=1}^{n}B_k(z)q_k^2 = \Lambda(z). \tag{58}$$

For $z = \beta_i :\ B(\beta_i) = 0,\ B_k(\beta_i) = \delta_{ki}B_k(\beta_k)$, hence

$$q_k^2 = 4\frac{\Lambda(\beta_k)}{B_k(\beta_k)} = 4\frac{\prod_{j=1}^{n}(\beta_k-\lambda_j)}{\prod_{j=1,j\neq k}^{n}(\beta_k-\beta_j)}$$

and then from (57) and (58)

$$\rho_j(\lambda) = \rho_j(\beta) - \frac{1}{4}\sum_{j=1}^{n}\frac{\partial\rho_j(\beta)}{\partial\beta_k}q_k^2. \tag{59}$$

Elliptic coordinates are separation coordinates for the following family of Benenti systems

$$\sigma(\lambda) + H_1\lambda^{n-1} + ... + H_n = -\frac{1}{2}B(\lambda)\mu^2,$$

where the family of elliptic separable potentials is given by $\sigma(\lambda) = \lambda^k,\ k \in \mathbb{Z}$, and thus $V^{(k)}(q) = F^k(q)V^{(0)}$, where F is given by (29) and (59).

Let us relate two separation curves for $n = 2$, namely

$$P(\lambda, h, \alpha) \equiv \sigma(\lambda) + h_1\lambda + h_2 - B^{-1}(\lambda)\lambda(\alpha_1\lambda + \alpha_2) = \frac{1}{2}\lambda\mu^2 \tag{60}$$

and

$$\widetilde{P}(\lambda, \widetilde{h}, \widetilde{\alpha}) \equiv -B(\lambda)\sigma(\lambda)\lambda^{-1} - B(\lambda)(\widetilde{\alpha}_1 + \widetilde{\alpha}_2\lambda^{-1}) + \widetilde{h}_1\lambda + \widetilde{h}_2 = -\frac{1}{2}B(\lambda)\mu^2 \tag{61}$$

where both families are now from Benenti class with $(\gamma_1, \gamma_2) = (\delta_1, \delta_2) = (1, 0)$ and with $R(\lambda) = -B(\lambda)\lambda^{-1} = -(\lambda-\beta_1)(\lambda-\beta_2)\lambda^{-1}$, while $\sigma(\lambda)$ is again assumed to be an arbitrary rational function of λ. For the extended Hènon-Heiles system, when $\sigma(\lambda) = \lambda^4$ in (60), in flat orthogonal coordinates from the previous example, an appropriate Hamiltonians are

$$\begin{aligned} h_1 = {} & \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + x_1^3 + \frac{1}{2}x_1x_2^2 + \frac{4(\beta_1+\beta_2)x_2^2 - 16\beta_1\beta_2x_1}{(4\beta_1^2+4\beta_1x_1-x_2^2)(4\beta_2^2+4\beta_2x_1-x_2^2)}\alpha_1 \\ & + \frac{4x_2^2+16\beta_1\beta_2}{(4\beta_1^2+4\beta_1x_1-x_2^2)(4\beta_2^2+4\beta_2x_1-x_2^2)}\alpha_2, \end{aligned}$$

$$\begin{aligned} h_2 = {} & \frac{1}{2}x_1y_2^2 - \frac{1}{2}x_2y_1y_2 - \frac{1}{4}x_1^2x_2^2 - \frac{1}{16}x_2^4 + \frac{x_2^4+4\beta_1\beta_2x_2^2}{(4\beta_1^2+4\beta_1x_1-x_2^2)(4\beta_2^2+4\beta_2x_1-x_2^2)}\alpha_1 \\ & + \frac{4x_1x_2^2+4(\beta_1+\beta_2)x_2^2}{(4\beta_1^2+4\beta_1x_1-x_2^2)(4\beta_2^2+4\beta_2x_1-x_2^2)}\alpha_2. \end{aligned}$$

We pass now to the respective system from the family (61). The transformation from Viète to flat orthogonal coordinates is given by (59)

$$\rho_1(\lambda) = -\beta_1 - \beta_2 + \frac{1}{4}q_1^2 + \frac{1}{4}q_2^2,\ \ \rho_2(\lambda) = \beta_1\beta_2 - \frac{1}{4}\beta_2 q_1^2 - \frac{1}{4}\beta_1 q_2^2.$$

The metric $\widetilde{G}$ and the Killing tensor $\widetilde{K}_2$ of this system are

$$\widetilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \ \widetilde{K}_2 = \begin{pmatrix} -\beta_2 + \frac{1}{4}q_2^2 & -\frac{1}{4}q_1 q_2 \\ -\frac{1}{4}q_1 q_2 & -\beta_1 + \frac{1}{4}q_1^2 \end{pmatrix}$$

so that the geodesic Hamiltonians of (61) are

$$\widetilde{E}_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2, \qquad \widetilde{E}_2 = \frac{1}{2}(-\beta_2 + \frac{1}{4}q_2^2)p_1^2 + \frac{1}{2}(-\beta_1 + \frac{1}{4}q_1^2)p_2^2 - \frac{1}{4}q_1 q_2 p_1 p_2.$$

The matrix $\widetilde{F}_\gamma$ becomes

$$\widetilde{F}_\gamma = \begin{pmatrix} \beta_1 + \beta_2 - \frac{1}{4}q_1^2 - \frac{1}{4}q_2^2 & 1 \\ -\beta_1\beta_2 + \frac{1}{4}\beta_2 q_1^2 + \frac{1}{4}\beta_1 q_2^2 & 0 \end{pmatrix}$$

and hence

$$\widetilde{h}_1 = \widetilde{E}_1 + \widetilde{V}_1^{(\sigma)} - \frac{1}{4}(q_1^2 + q_2^2)\widetilde{\alpha}_1 - \frac{\beta_2 q_1^2 + \beta_1 q_2^2}{4\beta_1\beta_2 - \beta_2 q_1^2 - \beta_1 q_2^2}\widetilde{\alpha}_2,$$

$$\widetilde{h}_2 = \widetilde{E}_2 + \widetilde{V}_2^{(\sigma)} + \frac{1}{4}(\beta_2 q_1^2 + \beta_1 q_2^2)\widetilde{\alpha}_1 + \frac{\beta_2^2 q_1^2 + \beta_1^2 q_2^2}{4\beta_1\beta_2 - \beta_2 q_1^2 - \beta_1 q_2^2}\widetilde{\alpha}_2,$$

where $\widetilde{V}^{(\sigma)} = -B(F_\gamma)F_\gamma^3 V^{(0)}$, so

$$\begin{aligned}\widetilde{V}_1^{(\sigma)} &= -\frac{1}{4}(\beta_1^3 q_1^2 + \beta_2^3 q_2^2) + \frac{1}{8}(\beta_1\beta_2 + \beta_1^2 + \beta_2^2)q_1^2 q_2^2 + \frac{3}{16}(\beta_1^2 q_1^4 + \beta_2^2 q_2^4) - \frac{3}{64}(2\beta_1 + \beta_2)q_1^4 q_2^2 \\ &\quad - \frac{3}{64}(\beta_1 + 2\beta_2)q_1^2 q_2^4 - \frac{3}{64}(\beta_1 q_1^6 + \beta_2 q_2^6) + \frac{1}{64}(q_1^6 q_2^2 + q_1^2 q_2^6) + \frac{3}{128}q_1^4 q_2^4 + \frac{1}{256}(q_1^8 + q_2^8),\end{aligned}$$

$$\begin{aligned}\widetilde{V}_2^{(\sigma)} &= \frac{1}{4}\beta_1\beta_2(\beta_2^2 q_1^2 + \beta_1^2 q_2^2) - \frac{1}{16}(\beta_1^3 + 2\beta_1\beta_2^2 + 2\beta_1\beta_2^2 + \beta_2^3)q_1^2 q_2^2 - \frac{3}{16}\beta_1\beta_2(\beta_1 q_1^4 + \beta_2 q_2^4) \\ &\quad + \frac{1}{32}(\beta_1^2 + \frac{5}{2}\beta_1\beta_2 + \beta_2^2)(q_1^4 q_2^2 + q_1^2 q_2^4)\frac{3}{64}\beta_1\beta_2(q_1^6 + q_2^6) - \frac{1}{256}(3\beta_1 + \beta_2)q_1^2 q_2^6 \\ &\quad - \frac{1}{256}(\beta_1 + 3\beta_2)q_1^6 q_2^2 - \frac{3}{256}(\beta_1 + \beta_2)q_1^4 q_2^4 - \frac{1}{256}(\beta_2 q_1^8 + \beta_1 q_2^8).\end{aligned}$$

According with (48), we have now

$$\widetilde{H} = R(F_\gamma)H = -B(F_\gamma)F_\gamma^{-1}H,$$

where

$$R(F_\gamma) = \begin{pmatrix} \beta_1 + \beta_2 + x_1 & -1 - 4\beta_1\beta_2 x_2^{-2} \\ -\frac{1}{4}x_2^2 - \beta_1\beta_2 & \beta_1 + \beta_2 - 4\beta_1\beta_2 x_1 x_2^{-2} \end{pmatrix}$$

while the reciprocal transformation (49) attains the form

$$dt = R(\widetilde{F}_\gamma^T)d\widetilde{t} = -\left(\frac{\partial\widetilde{h}}{\partial\widetilde{\alpha}}\right)^T d\widetilde{t} \tag{62}$$

where

$$R(\widetilde{F}_\gamma^T) = \begin{pmatrix} \frac{1}{4}(q_1^2 + q_2^2) & -\frac{1}{4}(\beta_2 q_1^2 + \beta_1 q_2^2) \\ (\beta_2 q_1^2 + \beta_1 q_2^2)(4\beta_1\beta_2 - \beta_2 q_1^2 - \beta_1 q_2^2)^{-1} & -(\beta_2^2 q_1^2 + \beta_1^2 q_2^2)(4\beta_1\beta_2 - \beta_2 q_1^2 - \beta_1 q_2^2)^{-1} \end{pmatrix}.$$

In (62) we used the inverse of formula (49) to avoid the complicated matrix $R^{-1}(F_\gamma)$. As before the transformation leaves the common level surface $M_{\alpha,\widetilde{\alpha}}$ invariant. Also, this time we have $\widetilde{P}(\lambda, \alpha_1, \alpha_2, a_1, a_2) = -B(\lambda)\lambda^{-1}P(\lambda, a_1, a_2, \alpha_1, \alpha_2)$. The point transformation between flat coordinates of both systems is

$$x_1 = -\beta_1 - \beta_2 + \frac{1}{4}q_1^2 + \frac{1}{4}q_2^2, \qquad x_2^2 = \beta_1 q_2^2 + \beta_2 q_1^2 - 4\beta_1\beta_2.$$

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